Exercise 1

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \exp(-\frac{1}{x})$ if $x > 0$, and $f(x) = 0$ if $x \leq 0$.

(1) Is f continuous at 0? Is it right differentiable at 0? Is it left differentiable at 0?

(2) Show that for every integer $n \geq 1$, there exists a polynomial $P_n(x)$ such that $f^{(n)}(x) = \frac{P_n(x)}{x^{2n}} \exp(-\frac{1}{x})$, and find a recurrence relation for P_n.

(3) Show that f is of class C^∞, namely that it has a derivative of any order. Can it be expressed as a series $f(x) = \sum b_n x^n$ of positive radius of convergence at the neighborhood of 0?

(4) Show that there exists a function $g : \mathbb{R} \to \mathbb{R}$ with the following properties:
   - g is of class C^∞,
   - $g(x) = 0$ if $|x| \geq 1$,
   - $g(x) > 0$ if $|x| < 1$.

Exercise 2

Let n be a positive integer number. Denote by E the vector space $\mathbb{R}_n[X]$ of real polynomials of degree $\leq n$ in one variable, and equip E with the scalar product

$$\langle P, Q \rangle = \int_0^1 P(t)Q(t)dt.$$

The norm of $P \in E$ is denoted by $\|P\|$. Recall that $\|P\|^2 = \langle P, P \rangle$.

Denote by $F \subset E$ the subspace generated by X, X^2, …, X^n. Our purpose is to determine the distance $d = d(1, F)$, which is by definition, the infimum of $\|1 - P\|$ for $P \in F$:

$$d(1, F) = \inf\{\|1 - P\| \mid P \in F\}.$$

Put

$$S(X) = \frac{(X-1)\dots(X-n)}{(X+1)\dots(X+n+1)}.$$

The partial fraction decomposition of S is of the form

$$S(X) = \frac{a_0}{x+1} + \dots + \frac{a_n}{X+n+1},$$

where a_0, …, a_n are real numbers.

(1) Compute a_0.

(2) Prove that the polynomial

$$T(X) = a_n X^n + \dots + a_0$$

is orthogonal to F, namely $\langle T, P \rangle = 0$ for every $P \in F$.

(3) Describe $F^\perp$, the set of polynomials orthogonal to F.

(4) Show that $d^2 = \frac{1}{a_0^2}\|T\|^2$.

(5) Compute d.

Exercise 3

A partition $(a_1, a_2, \dots, a_s)$ of a positive integer $n \geq 1$ is a finite decreasing sequence of integers $a_1 \geq a_2 \geq \cdots \geq a_s > 0$, called summands, such that $a_1 + \cdots + a_s = n$. The number of partitions of n is denoted by $p(n)$. For example, the partitions of 3 are (3); (2, 1); (1, 1, 1), and so $p(3) = 3$.

(1) Give the partitions of 4 and 5; what is the value of $p(4)$ and $p(5)$?

(2) Show that $p(n)$ is also the number of sequences of non-negative integers $(x_k)_{k=1}^{\infty}$ which verify $\sum_{k=1}^{\infty} kx_k = n$.

(3) Let $0 < t < 1$ be a real number. Show that the sequence $u_1, u_2, \dots,$ defined by
$$\forall\, m \geq 1\,,\ \ u_m := \prod_{k=1}^{m} \frac{1}{1-t^k}$$
is strictly increasing and convergent.

Define $f(t) := \prod_{k=1}^{\infty} \frac{1}{1-t^k}$.

(4) Prove that $f(t) = 1 + \sum_{n=1}^{\infty} p(n)t^n$.

Let $s, m \geq 1$ be positive integers. For an integer $n \geq 1$, denote by $q_{s,m}(n)$ the number of partitions of n into s distinct summands such that the maximum of the summands is m. In other words, $q_{s,m}$ is the number of partitions of n of the form $(a_1, a_2, \dots, a_s)$ with $a_1 = m > a_2 > \cdots > a_s$. For example, $q_{1,3}(3) = q_{2,4}(6) = 1$.

Denote by $q(n) = \sum_{s=1}^{\infty} \sum_{m=1}^{\infty} q_{s,m}(n)$ the number of partitions of n into distinct summands.

(5) What are the values of $q(7)$ and $q(8)$?

(6) Show that $q_{s,m}(n)$ is also the number of partitions of n into m summands of the form $(b_1, \dots, b_m)$ such that $b_1 = s$ and such that all the integers $1, \dots, s$ appear at least once among the summands $b_1, \dots, b_m$.

(7) Deduce that $q(n)$ is also the number of partitions of n such that if an integer $k \geq 2$ appears among the summands, then $k - 1$ appears as well.

(8) Prove that $q(n)$ is also the number of partitions of n into odd summands, i.e., the number of partitions of n of the form $(a_1, \dots, a_s)$, for an integer $s \geq 1$, such that $a_1, \dots, a_s$ are odd numbers.